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A MATHEMATICAL STUDY OF CRYSTAL SYMMETRY.

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CONTENTS

Introduction.

The modern classification of crystals is based primarily upon symmetry. The types of symmetry possible in crystals are now firmly established. As early as 1830, Hessel predicted the existence of the 32 crystal classes when representatives of only 17 of the 32 were known. Hessel's work was overlooked for many years, but was confirmed by later investigators. At the present time examples of all but one (the trigonal bipyramidal class) of the 32 possible crystal classes have been found, either among minerals or products of the laboratory.

Although the various types of crystal symmetry are now well established, there is a decided lack of uniformity in the manner of expressing the symmetry. Some authors emphasize planes of symmetry while others emphasize axes of symmetry. Many authorities disregard the center of symmetry. Some writers use rotatory-reflections as symmetry operations, while a few employ rotatory-inversions instead. There is, indeed, a marked difference of opinion as to what constitutes the true elements of symmetry.

This paper is presented with the conviction that a mathematical treatment of the subject will settle the disputed points and enable us

to determine what the true symmetry elements are. If any subject in the whole realm of science is capable of mathematical treatment, it would seem that crystal symmetry is, for in crystals we find Nature in her most mathematical mood.

One of the points of contention is the use of the center of symmetry. The center of symmetry was included by Bravais in his classic paper,[1] which was the first serious contribution to the study of crystal symmetry. But it has been contended by Fedorov, Groth, and others that the center of symmetry is not a true element of symmetry. By appeal to the mathematical theory of groups the writer has been able to settle this question definitely. The consideration of the center of symmetry led to a thorough study of all the other possible elements of symmetry.

Symmetry Operations

In a critical study of symmetry it has been found convenient to use the term *operation*, which may be defined as a movement of some kind which brings a polyhedron into coincidence either with itself or its mirror image. The only movements or operations that concern us here are those that leave the center of the figure unmoved.

In a discussion of crystal symmetry it is customary to deal only with faces of the general form, that is, with *hkl* faces or those which are neither parallel nor perpendicular to axes or planes of symmetry. A further advantage in our study is gained if we consider the operations necessary to derive all the faces of a general form from a single initial face. This method furnishes the key to the solution of the problem before us. Let us now consider the various symmetry operations possible in crystals.

Rotation about an Axis. Since the possible axes of symmetry in crystals are restricted to 2-fold, 3-fold, 4-fold, and 6-fold, the angles of rotation are limited to the crystallometric angles 60°, 90°, 120°, 180°, 240°, 270°, and 300° (360° gives identity). These operations may be designated by the following symbols: $a_{60°}$, $a_{90°}$, $a_{120°}$, $a_{180°}$, $a_{240°}$, $a_{270°}$, and $a_{300°}$. Throughout this article rotations are considered

[1] *Mémoire sur les polyèdres de forme symétrique, Jour. de Math.*, vol. 14, pp. 141–180, 1849. Reprinted in *Etudes Cristallographiques*, pp. xxi-lxii, Paris, 1866.

to be counter-clockwise. The operations are represented by Figs. 2–8.[2]

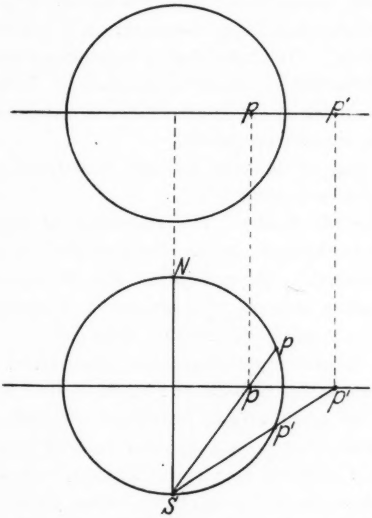

FIG. 1. Plan and elevation showing the method of plotting upper (p) and lower (p') faces on a stereographic projection.

Reflection in a Plane. This operation may be designated by the symbol p. It is represented by Fig. 9.

Wulff[3] has shown that all symmetry operations may be expressed in terms of planes of symmetry. Successive reflections in two planes of symmetry are equivalent to a rotation about an axis of symmetry.

[2] These and the following figures are stereographic projections. In these projections faces on the upper half of the crystal appear within the equatorial circle, while faces on the lower half of the crystal appear outside the equatorial circle. The accompanying figure (Fig. 1) shows the method of constructing the projections and the relation between the upper (p) and lower (p') faces. The angle Np equals the angle Sp'. In this method both projections are made from the south pole; in the usual method (first employed by Gadolin) projections for faces on the upper half of the crystal are made from the south pole and projections for faces on the lower half of the crystal are made from the north pole.

[3] *Zeit. f. Kryst. u. Min.*, vol. 27, pp. 556–558, 1897.

While this is true, it should be emphasized that the two virtual[4] planes of symmetry are situated at crystallometric angles to each other, which really means that rotation is the effective operation.

Becke,[5] on the other hand, has shown that it is possible to disregard planes of symmetry. He states that a rotatory-inversion[6] of 180° is equivalent to a reflection in a plane of symmetry. Now it is advisable at times to use a rotatory-inversion, but it is not necessary here, for reflection is a far simpler operation.

The simplest way of deriving an $(h\bar{k}l)$ face from the initial (hkl) face is by reflection in a plane (010).

Inversion about the Center. The operation of passing from the initial face (hkl) to the opposite parallel face $(\bar{h}\bar{k}\bar{l})$ is called *inversion.* It may be represented by the symbol *c*. Fig. 10 illustrates inversion.

The corresponding element of symmetry is a center of symmetry designated by *C*, a symbol first used by Bravais.[7]

Fedorov,[8] the Russian crystallographer, maintained that the center of symmetry is not a true element of symmetry, but simply the effect of a rotation of 180° about an axis, combined with reflection in a plane normal to the axis, that is, a particular case of rotatory-reflection. Fedorov has been followed by Groth,[9] Tutton,[10] Swartz,[11] Jellinek,[12] Oebbeke and Weinschenk,[13] and others. One gains the impression that some authors use the center of symmetry simply because of the difficulty involved in the operation of rotatory-reflection. Dana,[14] for example, says, " This method [he refers to rotatory-reflection] is not followed here since, though having certain theoretical advantages, it is likely to confuse the student meeting the problems of crystallog-

[4] This term is used by Jaeger, *Lectures on the Principles of Symmetry*, 1st ed., p. 30, Amsterdam, 1917.

[5] *Zeit. f. Kryst. u. Min.*, vol. 25, pp. 73–78, 1896.

[6] This term is explained on page 168.

[7] Loc. cit., p. 141.

[8] *Verhandl. Russ. Min Ges.*, vol. 25, pp. 5–7, 1899. *Zeit. f. Kryst. u. Min.*, vol. 21, p. 586, 1893.

[9] *Lehrbuch der Physikalischen Krystallographie*, 4th ed., p. 329, Leipzig, 1905.

[10] *Crystallography and Practical Crystal Measurement*, 2nd ed., vol. 1, p. 128, p. 135, London, 1922.

[11] *Bull. Geol. Soc. Amer.*, vol. 20, pp. 384–5, 1909.

[12] *Lehrbuch der Physikalischen Chemie*, vol. 2, p. 414, Stuttgart, 1915.

[13] Von Kobell's *Lehrbuch der Mineralogie*, 6th ed., p. 17, Leipzig, 1899.

[14] *Text Book of Mineralogy*, 2nd ed., p. 10, 1898.

FIG. 2. $a_{60°}$

FIG. 3. $a_{90°}$

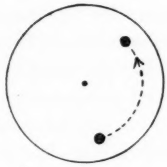

FIG. 4. $a_{120°}$

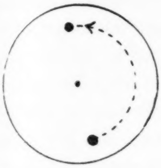

FIG. 5. $a_{180°}$

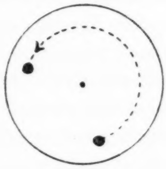

FIG. 6. $a_{240°}$

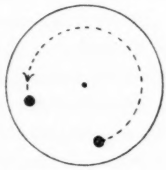

FIG. 7. $a_{270°}$

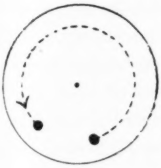

FIG. 8. $a_{300°}$

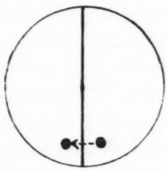

FIG. 9. p

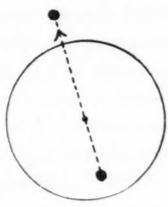

FIG. 10. c

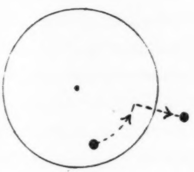

FIG. 11. $ap_{60°}$

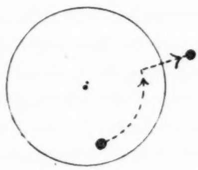

FIG. 12. $ap_{90°}$

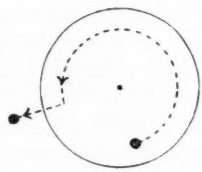

FIG. 13. $ap_{270°}$

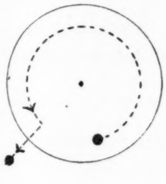

FIG. 14. $ap_{300°}$

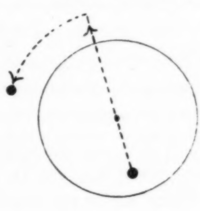

FIG. 15. $ca_{60°}$

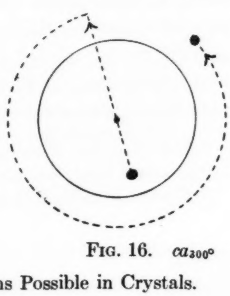

FIG. 16. $ca_{300°}$

FIGS. 2–16. The Simple Symmetry Operations Possible in Crystals.

raphy for the first time." While the idea of rotatory-reflection may be somewhat difficult for the beginning student to grasp, the validity of the center of symmetry as an element of symmetry should be decided solely upon its merits.

Some of those who discard the center of symmetry fail to realize that the two-fold axis and plane of composite symmetry may have any position whatever in the crystal. This point has been emphasized by Marshall.[15] An axis or plane of symmetry not fixed in direction is obviously an absurdity. Any line through the geometric center of the crystal is a 2-fold axis of rotatory-reflection, AP_2. Hence the symmetry according to this method must be expressed by the symbol $\infty\ AP_2$ or its equivalent. It is true that an $(\bar{h}\bar{k}\bar{l})$ face may be derived from the initial (*hkl*) face by a rotatory-reflection of 180°, but when we come to the elements of symmetry we have to write $\infty\ AP_2$ in order to include all the possible symmetry operations. But an opposite, parallel face is derived from the initial face by a single operation and not by an infinitude of operations. This operation has been called *inversion*. Since the operation takes place about a point which is the geometric center of the crystal, the crystal is said to be symmetrical to a center. The intersection of an infinite number of lines (2-fold axes of rotatory-reflection) is a point, which is called the center of symmetry. Hence *the center of symmetry is a true element of symmetry.*

Another argument in favor of considering the center of symmetry as an element of symmetry is the existence of point-twinning as it has been termed by Evans.[16] This type of twinning was first mentioned by Bravais.[17] It has been overlooked by most writers, but has been emphasized by Evans. The best examples of point-twinning are the "Brazilian" twins of quartz. As the twin-crystal apparently has a center of symmetry, the twinning may be described as inversion-twinning or point-twinning. The two parts of the twin are symmetrical to a point. These quartz twins are usually described as being twinned on the $(11\bar{2}0)$ face, but in point-twinning every plane normal to an axis of even-fold symmetry has the characters of a twin-plane, and as Evans indicates, point-twinning is more important than plane-twinning that accompanies it.

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[15] *Proc. Roy. Soc. Edinburgh*, vol. 25, I, pp. 383–4, 1904.

[16] *Mineral. Mag.*, vol. 15, p. 393, 1909; *ibid.* vol. 18, pp. 224–243, 1918.

[17] *Etudes Cristallographiques*, IIIme Partie pp. 258–9, Paris, 1866.

Eleven classes out of the 32 have a center of symmetry.

Rotatory-reflection with respect to a Plane and an Axis of Rotation normal thereto. There are operations which cannot be referred to either rotations, reflections, or inversions. On crystals of the tetragonal bisphenoidal class, for example, there are four faces in the general form. The initial face is (hkl); the ($\bar{h}\bar{k}l$) face is produced by a rotation of 180°; the simplest way to obtain the other two faces, ($\bar{k}h\bar{l}$) and ($k\bar{h}\bar{l}$), is to combine a rotation of 90° and 270° respectively with a reflection in a plane normal to the axis of rotation. The combined effect of rotation and reflection is considered to be a single operation, called for convenience *rotatory-reflection.* It might be well to use the term *rotoflection* for this operation. The symbols used are $ap_{90°}$ (Fig. 12) and $ap_{270°}$ (Fig. 13). Compound symbols are used here since rotation and reflection taken together are effective operations. Besides these there are two other rotatory-reflections possible in crystals, $ap_{60°}$ (Fig. 11) ($hk\bar{i}l$ to $\bar{k}i\bar{h}\bar{l}$) and $ap_{300°}$ (Fig. 14) ($hk\bar{i}l$ to $i\bar{h}\bar{k}\bar{l}$). As has been shown on page 166 a rotatory-reflection of 180° is equivalent to a center of symmetry, and the point was made that the latter term must be used instead.

Although he discussed it, Bravais[18] failed to include the composite axis-plane of symmetry and as a consequence he deduced only 31 out of the 32 possible crystal classes. The rhombohedral and hexagonal scalenohedral classes were included by Bravais since they may be obtained by adding a center of symmetry to two other classes of the hexagonal system, but the tetragonal bisphenoidal class, which was the one omitted by Bravais, has no center of symmetry.

Although the tetragonal bisphenoidal class was included by Hessel[19] and by Gadolin[20] in their list of the 32 classes, P. Curie,[21] the husband of Madame Curie, was the first to give full recognition to the composite plane and axis of 4-fold symmetry, under the expression "plans de symétrie alterne."

Fedorov recognized rotatory-reflection under the term "composite

[18] *Etudes Cristallographiques,* IIme Partie, p. 229, 1866.

[19] *Krystallometrie, oder Krystallonomie und Krystallographie* (reprinted in Ostwald's *Klassiker der Exakten Wissenschaften,* Nos. 88 and 89).

[20] *Acta Societatis Scientiarum Fennicae,* vol. 9, p. 28, 1871 (reprinted in German in Ostwald's *Klassiker der Exakten Wissenschaften,* no. 75.)

[21] *Bull. Franc. Soc. Min.,* vol. 7, pp. 453–5, 1884.

symmetry" ("zusammengesetzte Symmetrie"), but, as will be seen later, there are two types of composite symmetry, rotatory-reflection and rotatory-inversion.

It is absolutely necessary to recognize rotatory-reflections as symmetry operations. If rotatory-reflections are not recognized then there should be 31 classes instead of 32. In that case the tetragonal bisphenoidal class must be included in the sphenoidal class of the monoclinic system, but there are four faces in the general form of the first mentioned class and only two in the latter. In order to account for all the faces it is necessary to include rotatory-reflections. If rotatory-reflections are used in this class, it is necessary to use them in other classes also. As will be shown later, it is necessary to use rotatory-reflections in 11 classes in order that all the faces of the general form may be derived from the initial (*hkl*) face.

Rotatory-inversion with respect to a Center and an Axis of Rotation. Besides rotations, reflections, inversions, and rotatory-reflections there is still another type of symmetry operation effective in crystal symmetry. If we consider, for example, the dihexagonal bipyramidal class, there are 24 faces in the general form. Twenty-two out of 24 of the faces may be derived from the initial face ($hk\bar{i}l$) by rotations, reflections, and rotatory-reflections (including with these identity). There remain two faces ($k\bar{i}h\bar{l}$) and ($\bar{i}hk\bar{l}$) which are unaccounted for. The simplest method of deriving these two faces from the ($hk\bar{i}l$) face is by combining rotations of 60° and 300° respectively around an axis with inversion about the center of the crystal. The combined effect of rotation and inversion is considered to be a single operation, called for convenience *rotatory-inversion*. It might be well to use the term *rotoversion* for this operation. The two possible cases in crystals may be designated by the symbols $ca_{60°}$ (Fig. 15) and $ca_{300°}$ (Fig. 16).

For angles of rotation of 90° and 270° rotatory-inversions are equivalent to rotatory-reflections, and in these cases the latter term is preferred. A rotatory-inversion of 180° is equivalent to a reflection in a plane of symmetry.

The idea of rotatory-inversion was recognized by Möbius[22] as early as 1851, but has been overlooked or disregarded by most crystallographers. The following writers, however, have made use of rota-

[22] *Ber. der Königl. Sachs. Gesell. der Wissen.*, 1851, p. 349.

tory-inversions: Gadolin,[23] Becke,[24] Liebisch,[25] Hilton,[26] Evans,[27] Saurel,[28] Tertsch,[29] Wyckoff,[30] Sen,[31] and Evans and Davies.[32]

Evans[33] uses the term " inverse axis of symmetry " for an axis of rotatory-inversion.

Hilton[34] has placed especial emphasis upon rotatory-inversions. Hilton states: " . . . it is absolutely essential for crystal structure purposes to take the rotation and rotatory-inversion " as fundamental operations, but on analysis, all that this means is that, by discarding axes of rotatory-reflection and using axes of rotatory-inversion, one obtains a satisfactory division of the 32 crystal classes into seven systems. The trigonal bipyramidal (No. 21) and ditrigonal bipyramidal (No. 22) classes then have a 6-fold principal axis and are included in the hexagonal system, while the five crystal classes with a 3-fold principal axis are placed in the rhombohedral system.

According to Hilton, both the rhombohedral (No. 17) and scalenohedral (No. 20) classes have a 3-fold rotatory-inversion axis. But it should be emphasized that in the case of both n-fold rotatory-inversion axes and n-fold rotatory-reflection axes n must be an even number. This will be brought out later in the paper. The principal axis in the rhombohedral and scalenohedral classes is a 6-fold axis of rotatory-reflection; therefore Hilton's mnemonic distinction between the hexagonal and rhombohedral system (or subsystems if one prefers six systems) fails.

Without exception the authors who use rotatory-inversions discard rotatory-reflections. The method of deriving every face of the general form from the initial face independently of the other faces proves that *both rotatory-reflections and rotatory-inversions are symmetry operations.* One is as fundamental as the other.

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[23] Loc. cit., pp. 15–19.

[24] *Zeit. f. Kryst. u. Min.*, vol. 25, pp. 73–78, 1896.

[25] *Grundriss der Physikalischen Krystallographie*, pp. 66–68, Leipzig, 1896.

[26] *Mineral. Mag.*, vol. 14, pp. 261–3, 1907.

[27] *Mineral. Mag.*, vol. 15, pp. 398–400, 1912.

[28] *Zeit. f. Kryst. u. Min.*, vol. 50, pp. 1–5, 1912.

[29] *Centralblatt f. Min., Geol., u. Pal.*, 1916, pp. 171–180.

[30] *Am. Jour. Sci.* [5th series] vol. 6, pp. 288–290, 1923.

[31] *Pan-American Geologist*, vol. 42, pp. 331–4, 1924.

[32] *Elementary Crystallography*, pp. 122–131, London, 1924.

[33] *Nature*, vol. 113, p. 81, 1924.

[34] *Mineral. Mag.*, vol. 19, pp. 319–322, 1922.

The other possible composite symmetry operation, reflection-inversion, is equivalent to a simple rotation of 180°, which has already been considered.

Identical Operation. As the initial face is restored to its original position by n rotations of $360°/n$ about an n-fold axis of rotation, rotatory-reflection, or rotatory-inversion, and by two reflections (in the same plane) or two inversions, every crystal is considered to have the *identical operation.* This device is for the sake of mathematical completeness. For an n-fold axis there are n operations, provided the identical operation is used. A crystal in which only the identical operation is effective has no symmetry.

The symbol of the identical operation is 1.

The symmetry operations possible in crystals, then, are as follows:

Identical operation, 1.
Rotations, $a_{60°}$, $a_{90°}$, $a_{120°}$, $a_{180°}$, $a_{240°}$, $a_{270°}$, $a_{300°}$.
Reflection, p.
Inversion, c.
Rotatory-reflections, $ap_{60°}$, $ap_{90°}$, $ap_{270°}$, $ap_{300°}$.
Rotatory-inversions, $ca_{60°}$, $ca_{300°}$.

These operations are illustrated by Figs. 2–16 of page 165.

Rotations are usually called "operations of the first sort" and the other operations, "operations of the second sort," but too much emphasis has been placed upon this distinction.

The above tabulation suggests a new definition of symmetry operation. *Symmetry operations may be defined as movements by means of which each and every face of the general form of a crystal may be derived directly from an arbitrarily selected face.*

Symmetry Elements.

The symmetry operations take place about an axis, plane, center, or about two of these simultaneously. These are collectively called elements of symmetry, a term first used by Bravais.[35]

What are the possible elements of symmetry? Since the symmetry operations have been determined, this is an easy question to answer.

A consideration of symmetry operations shows that some of them may be obtained by repeating another operation a certain number of times. For example, $a_{180°}$ is obtained by repeating $a_{90°}$, and $a_{270°}$ by

[35] Loc. cit., p. lx.

repeating $a_{90°}$ twice. These are called *powers of operations.* $a_{180°}$ is the second power of $a_{90°}$ $[(a_{90°})^2 = a_{180°}]$ and $a_{270°}$ the third power $[(a_{90°})^3 = a_{90°}]$. The axis of 4-fold symmetry (A_4) implies the four following operations: 1, $a_{90°}$, $a_{180°}$, $a_{270°}$. In a similar manner A_2, a 2-fold axis of symmetry, implies 1 and $a_{180°}$; A_3, a 3-fold axis, implies 1, $a_{120°}$, and $a_{240°}$; and A_6, a 6-fold axis, implies 1, $a_{60°}$, $a_{120°}$, $a_{180°}$, $a_{240°}$, and $a_{300°}$.

The second powers of a reflection and inversion are each equivalent to the identical operation.

If we repeat the rotatory-reflection $ap_{90°}$ we obtain the following: 1, $ap_{90°}$, $a_{180°}$, $ap_{270°}$. These four operations are included or implied in the use of the 4-fold axis of rotatory-reflection, for which a convenient symbol is AP_4. Similarly, a 6-fold axis of rotatory-reflection (AP_6) includes the following operations which are the six powers of $ap_{60°}$: 1, $ap_{60°}$, $a_{120°}$, c, $a_{240°}$, $ap_{300°}$.

The 6-fold axis of rotatory-inversion (CA_6) includes the following operations which are the six powers of $ca_{60°}$: 1, $ca_{60°}$, $a_{120°}$, p, $a_{240°}$, $ca_{300°}$. The axis of rotatory-inversion is exactly analogous to the axis of rotatory-reflection. The third power of the former is a reflection, while the third power of the latter is inversion. The even powers in each case are ordinary rotations.

A 4-fold axis of rotatory-inversion is the equivalent of a 4-fold axis of rotatory-reflection.

The following tabulation shows the relation between the symmetry elements[36] and symmetry operations:

A_2 : 1, $a_{180°}$.

A_3 : 1, $a_{120°}$, $a_{240°}$.

A_4 : 1, $a_{90°}$, $a_{180°}$, $a_{270°}$.

A_6 : 1, $a_{60°}$, $a_{120°}$, $a_{180°}$, $a_{240°}$, $a_{300°}$.

P : 1, p.

C : 1, c.

AP_4 : 1, $ap_{90°}$, $a_{180°}$, $ap_{270°}$.

AP_6 : 1, $ap_{60°}$, $a_{120°}$, c, $a_{240°}$, $ap_{300°}$.

CA_6 : 1, $ca_{60°}$, $a_{120°}$, p, $a_{240°}$, $ca_{300°}$.

[36] The symbols used for the symmetry elements, with the exception of CA_6, were given in the writer's *Introduction to the Study of Minerals*, 1st ed., New York, 1912.

It will be noted that for an n-fold axis there are n operations involved; also that for an n-fold axis the angle of rotation is always $360°/n$. For the axes AP_n and CA_n, n is always even.

The above tabulation shows that the elements of symmetry of a crystal express in a condensed form the symmetry operations. Symmetry may be expressed either by symmetry elements or symmetry operations. As the elements of symmetry may be indicated in a more abbreviated form they are usually to be preferred.

It should be noted that AP_4 includes A_2, and AP_6 includes A_3 and C, and also that CA_6 includes A_3 and P. While these relations are obvious, it has been thought best to include (C) and (P) in parentheses as indicated.

The elements of symmetry occur in different positions on different crystals. A plane of symmetry (P) may have as many as 13 different positions on various crystals. An axis of 2-fold symmetry (A_2) may also occur in 13 different positions. An axis of 3-fold symmetry (A_3) and composite axis-plane of 6-fold symmetry (AP_6) each may have five different positions. An axis of 4-fold symmetry (A_4) and composite axis-plane of 4-fold symmetry (AP_4) each may have three different positions. For an axis of 6-fold symmetry (A_6) and composite axis-center of 6-fold symmetry (CA_6) there is only one position possible.

The positions of the various symmetry elements are best shown by means of stereographic projections (see Figs. 17–48).

Upon analysis it is found that there are in all 64 different symmetry operations involved. All of these are represented by faces of the general form of the minerals beryl (dihexagonal bipyramidal class) and garnet (hexoctahedral class). There are 24 faces on the former and 48 on the latter, but 8 faces or operations are common to the two, which gives us 64. ($24 + 48 - 8 = 64$). The eight operations common to these two are the operations of the rhombic bipyramidal (No. 8) class.

The accompanying table (p. 173–4) shows the symmetry operations corresponding to the various faces of the general form for all possible crystals. These are listed in two groups, one with four indices for hexagonal crystals, the other with three indices for non-hexagonal crystals.

These 64 operations occur combined on crystals in various ways.

Only certain operations are compatible and can occur together. It has been found that there are 31 possible combinations of the various symmetry operations or elements. These 31 together with crystals devoid of symmetry (class 1) constitute the 32 crystal classes.

OPERATIONS CORRESPONDING TO FACES OF THE GENERAL FORM.

Systems Other than Hexagonal.

Upper Faces.

1st octant	2nd octant	3rd octant	4th octant
(hkl) 1	$(\bar{h}kl)$ p'''	$(\bar{h}\bar{k}l)$ $a'_{180°}$	$(h\bar{k}l)$ p''
(hlk) p^{VI}	$(\bar{h}lk)$ $a'''_{180°}$	$(\bar{h}\bar{l}k)$ $ap''_{90°}$	$(h\bar{l}k)$ $a''_{90°}$
(khl) p^{IV}	$(\bar{k}hl)$ $a'_{90°}$	$(\bar{k}\bar{h}l)$ p^{V}	$(k\bar{h}l)$ $a'_{270°}$
(klh) $a'_{240°}$	$(\bar{k}lh)$ $ap'''_{60°}$	$(\bar{k}\bar{l}h)$ $a^{IV}_{120°}$	$(k\bar{l}h)$ $ap''_{300°}$
(lhk) $a'_{120°}$	$(\bar{l}hk)$ $ap^{IV}_{60°}$	$(\bar{l}\bar{h}\bar{k})$ $a''_{240°}$	$(l\bar{h}k)$ $ap'''_{300°}$
(lkh) p^{VII}	$(\bar{l}kh)$ $a'''_{90°}$	$(\bar{l}\bar{k}h)$ $ap'''_{270°}$	$(l\bar{k}h)$ $a'''_{180°}$

Lower Faces.

5th octant	6th octant	7th octant	8th octant
$(hk\bar{l})$ p'	$(\bar{h}k\bar{l})$ $a'''_{180°}$	$(\bar{h}\bar{k}\bar{l})$ c	$(h\bar{k}\bar{l})$ $a''_{180°}$
$(hl\bar{k})$ $a''_{270°}$	$(\bar{h}l\bar{k})$ $ap''_{270°}$	$(\bar{h}\bar{l}\bar{k})$ $a^{VII}_{180°}$	$(h\bar{l}\bar{k})$ p^{VIII}
$(kh\bar{l})$ $a^{IV}_{180°}$	$(\bar{k}h\bar{l})$ $ap'_{90°}$	$(\bar{k}\bar{h}\bar{l})$ $a^{V}_{180°}$	$(k\bar{h}\bar{l})$ $ap'_{270°}$
$(kl\bar{h})$ $ap^{IV}_{300°}$	$(\bar{k}l\bar{h})$ $a''_{120°}$	$(\bar{k}\bar{l}\bar{h})$ $ap'_{60°}$	$(k\bar{l}\bar{h})$ $a'''_{240°}$
$(lh\bar{k})$ $ap''_{60°}$	$(\bar{l}h\bar{k})$ $a^{IV}_{120°}$	$(\bar{l}\bar{h}\bar{k})$ $ap'_{300°}$	$(l\bar{h}\bar{k})$ $a^{IV}_{240°}$
$(lk\bar{h})$ $a'''_{90°}$	$(\bar{l}k\bar{h})$ p^{IX}	$(\bar{l}\bar{k}\bar{h})$ $a^{VI}_{180°}$	$(l\bar{k}\bar{h})$ $ap'''_{90°}$

(Continued on next page)

Hexagonal System.

(Four Axes of Reference.)

Upper Faces.

$(hk\bar{i}l)$ 1	$(kh\bar{i}l)$ p^{V}	$(\bar{k}i\bar{h}l)$ $a'_{60°}$	$(\bar{h}i\bar{k}l)$ p'''	$(\bar{i}hkl)$ $a'_{120°}$	$(\bar{i}khl)$ p^{VI}
$(\bar{h}\bar{k}il)$ $a'_{180°}$	$(\bar{k}\bar{h}il)$ p^{IV}	$(k\bar{i}hl)$ $a'_{240°}$	$(h\bar{i}kl)$ p^{VII}	$(i\bar{h}\bar{k}l)$ $a'_{300°}$	$(i\bar{k}\bar{h}l)$ p''

Lower Faces.

$(hk\bar{i}\bar{l})$ p'	$(kh\bar{i}\bar{l})$ $a^{\mathrm{V}}_{180°}$	$(\bar{k}i\bar{h}\bar{l})$ $ap'_{60°}$	$(\bar{h}i\bar{k}\bar{l})$ $a'''_{180°}$	$(\bar{i}hk\bar{l})$ $ca'_{300°}$	$(\bar{i}kh\bar{l})$ $a^{\mathrm{VI}}_{180°}$
$(\bar{h}\bar{k}i\bar{l})$ c	$(\bar{k}\bar{h}i\bar{l})$ $a^{\mathrm{IV}}_{180°}$	$(k\bar{i}h\bar{l})$ $ca'_{60°}$	$(h\bar{i}k\bar{l})$ $a^{\mathrm{VII}}_{180°}$	$(i\bar{h}\bar{k}\bar{l})$ $ap'_{300°}$	$(i\bar{k}\bar{h}l)$ $a''_{180°}$

While there is one gap in the 32 classes to be filled, it should be remarked that all the possible symmetry operations and symmetry elements are represented by faces of the general form of the minerals beryl (dihexagonal bipyramidal class) and garnet (hexoctahedral class).

Application of the Theory of Groups to Crystal Symmetry.

Symmetry operations may be conveniently studied by means of the *theory of groups.* The group concept pervades the whole realm of mathematics, but as far as the writer knows, group theory has had no application outside the field of mathematics except in this one instance.

The theory of groups applies to a closed system of operations which have definite laws of combination. *A series of operations is said to form a group, provided the product of any two of them is equivalent to another operation of the series, and provided the inverse of any operation is also a member of the series.* Here " product " means the result of one operation followed by another. The identical operation is a member of every group since the product of any operation by its inverse gives identity. Since the above conditions are fulfilled, the symmetry operations of any crystal form a group. Let us take as an example the prismatic class of the monoclinic system with the sym-

metry elements: $A_2 \cdot P \cdot C$. Analyzing this we have the four operations: $\{1, a_{180°}, p, c\}$. The "product" of any two of these operations is another operation of the series. ($a_{180°} \cdot p = c$; $a_{180°} \cdot c = p$; $p \cdot c = a_{180°}$).

If the operations of any group are written down in a horizontal row and in a vertical column, their products may be shown in what is called a multiplication table. The "multiplication table" of the group $A_2 \cdot P \cdot C$ is as follows:

1	$a_{180°}$	p	c
$a_{180°}$	1	c	p
p	c	1	$a_{180°}$
c	p	$a_{180°}$	1

Each operation appears once in each row and once in each column. In this particular case the product of any two operations is *permutable*; that is, the result is the same regardless of the order ($a_{180°} \cdot p = p \cdot a_{180°} = c$).

This, however, is not always the case. For example, for the group $A_3 \cdot 3P$ the "multiplication table" is as follows:

1	$a'_{120°}$	$a'_{240°}$	p''	p'''	p^{IV}
$a'_{120°}$	$a'_{240°}$	1	p^{IV}	p''	p'''
$a'_{240°}$	1	$a'_{120°}$	p'''	p^{IV}	p''
p''	p'''	p^{IV}	1	$a'_{120°}$	$a'_{240°}$
p'''	p^{IV}	p''	$a'_{240°}$	1	$a'_{120°}$
p^{IV}	p''	p'''	$a'_{120°}$	$a'_{240°}$	1

Here $a'_{120^\circ} \cdot p'' = p^{IV}$, but $p'' \cdot a'_{120^\circ} = p'''$. These operations are not permutable.

It is also possible to construct " multiplication tables " for the other groups of crystallography.

Each of the 32 crystal classes constitutes a group. There are as many operations in a group as there are faces in the general form, and the number of operations in a group defines the *order of the group.* In geometrical crystallography there is one group of order 1, three of order 2, one of order 3 (the only prime group), five of order 4, five of order 6, five of order 8, six of order 12, one of order 16, four of order 24, and one of order 48.

Since a finite number of operations is involved, these groups are called *finite groups*; in crystal structure theory we encounter translations, screw-axes, and glide-planes, which, combined with the operations of the finite groups, constitute infinite groups of movements.

The finite groups of geometrical crystallography are also known as *point groups*, since the operations concerned all leave one point (the center of the figure) unmoved.

A group which consists entirely of the various powers of an operation is called a *cyclic group.* In this case each group may be represented by a single symmetry element. There are nine cyclic groups possible in crystallography (Classes Nos. 2, 3, 4, 9, 10, 16, 17, 21, and 23). They are represented by the tabulation on page 171. Of these only the first four are listed as cyclic by Schoenflies,[37] who apparently does not recognize cyclic groups of " the second sort."

The theory of groups has been applied to the study of crystal symmetry by Minnigerode,[38] Schoenflies,[39] Hilton,[40] Bouasse,[41] and Jaeger[42] with valuable results, but the method used by the present writer differs from the one employed by these writers, in that the symmetry operation necessary to obtain each face of a general form from the initial face is used. Schoenflies, Hilton, and all others who have employed group theory in crystallography make use of " generating "

[37] *Krystallsysteme und Krystallstructur*, Leipzig, 1891.

[38] *Neues Jahrb. f. Min. Geol. u. Pal.*, Beil. Bd. V, pp. 143–166, 1887.

[39] Loc. cit.

[40] *Mathematical Crystallography*, Oxford, 1903.

[41] *Cours de Physique*, 6 me. partie (Étude des Symétries), Paris, 1909.

[42] *Lectures on the Principle of Symmetry*, 1st ed., Amsterdam, 1917.

operations. Part of the operations of the non-cyclic groups are obtained by multiplying some of the operations of the group by an independent operation. These operations are used to "generate" the group. To illustrate, let us take the rhombic bipyramidal class or group. This group is represented by Schoenflies as follows (my symbols are used):

$$\left\{\begin{matrix} 1, & a'_{180^\circ}, & a''_{180^\circ}, & a'''_{180^\circ} \\ p', & p'\cdot a'_{180^\circ}, & p'\cdot a''_{180^\circ}, & p'\cdot a'''_{180^\circ} \end{matrix}\right\}$$

There are many different ways of representing this group. Instead of p' we might use either p'', p''', or c as one of the generating operations. The eight operations of the group are included in such an expression as the above, but they are not all expressed in terms of the operations performed on the initial face.

The preferable method is to express all eight operations independently of each other thus:

$$\{1,\ a'_{180^\circ},\ a''_{180^\circ},\ a'''_{180^\circ},\ p',\ p'',\ p''',\ c\}$$

This method brings out its relation to other groups which is not true of the method of generating operations. The use of generating operations usually conceals some of the symmetry relations of crystals. In stating the symmetry of a crystal class, it seems not unreasonable to include all the symmetry elements and not simply a part of them.

It is self-evident that a group of order n may be represented by n distinct symbols which uniquely define it. It is not necessary nor always advisable to express some of the operations as products of the others. For convenience, rotatory-reflections and rotatory-inversions are indicated in this paper by the compound symbols ap_{x° and ca_{x° respectively. Each of these symbols stands for a single operation; they do not express products of operation (expressed as products the operations would be $a_{x^\circ}\cdot p$ and $c\cdot a_{x^\circ}$ respectively). They do not necessarily represent products of operations of the group of which they are members. As an illustration, let us consider class 17, a cyclic group with the symmetry AP_6. In this class reflection (p) is not an operation of the group; so that $a_{60^\circ}\cdot p$ expressed as a product cannot possibly represent an operation of the group. However, ap_{60° expressed as a single operation is an operation of the group.

Similarly in class 21, a cyclic group with the symmetry CA_6, inversion (c) is not an operation of group; so that $c \cdot a_{60°}$ expressed as a product cannot possibly represent an operation of this group. But $ca_{60°}$ expressed as a single operation is an operation of the group.

Another difference between the expression $a_{x°} \cdot p$ and $ap_{x°}$ is this: in the former case the order of the two operations is indicated; the latter case represents a single operation of passing from one face to another without any other specifications.

The definition of a group states; " A series of operations is said to form a group . . . etc." Now it would seem necessary to enumerate the n operations of the group before anything further is said about the group, but, by the method of generating operations used exclusively for groups of " the second sort " by Schoenflies and Hilton, some operations are expressed as products. The fact that the product of two operations of the group is another operation of the group is the essence of group theory, but this property need not be anticipated in stating the group.

Although the method of generating operations may be useful at times in crystallography, the point is brought out here that the method of deriving every face of the general form from the initial face by a distinct operation enables us to decide just what are the true elements of symmetry.

Symmetry Elements and Operations in the Various Crystal Classes.

In the following pages the operations of each crystal class or group are represented by distinct symbols (the symmetry operations of p. 170 with appropriate marks to designate the position of the symmetry element as shown in the diagrams), and the Miller symbol of the particular face which is the result of performing a given operation is indicated.

The names used for the various crystal classes are the names of the general form in each case, except that in the class without any symmetry the name asymmetric is preferred to pediad (or pedial), a name derived from the general form pedion.

The order of arrangement of the crystal classes is the same as that of Groth's *Physikalische Krystallographie*, 4th ed., pp. 335–337, Leipzig, 1905, except that the trigonal bipyramidal class is placed

after the hexagonal (or ditrigonal) scalenohedral class. Groth's numbers 19, 20, and 21 become respectively my numbers 21, 19 and 20. This change has been made so that the trigonal pyramidal, rhombohedral, trigonal trapezohedral, ditrigonal pyramidal, and hexagonal scalenohedral classes (Nos. 16–20 inclusive), which constitute the rhombohedral subsystem (or system), may be together. The trigonal bipyramidal and ditrigonal bipyramidal classes belong in the hexagonal subsystem (or system) and not in the rhombohedral or trigonal subsystem (or system).

TRICLINIC SYSTEM

1. Asymmetric (or Pediad) Class (No symmetry).

1 operation: 1 (identity)
(hkl)

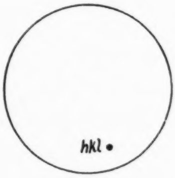

Fig. 17. Stereographic Projection of the General Form of Class 1.

2. Pinakoidal Class, C.

2 operations: 1 , c
(hkl) $(\bar{h}\bar{k}\bar{l})$

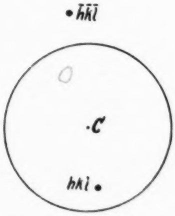

Fig. 18. Class 2.

MONOCLINIC SYSTEM

3. Sphenoidal Class, A_2.

2 operations: 1 , a_{180° .
(hkl) $(\bar{h}k\bar{l})$

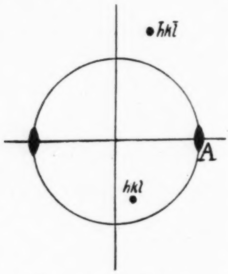

Fig. 19. Class 3.

4. Domatic Class, P.

2 operations: 1 , p .
(hkl) $(h\bar{k}l)$

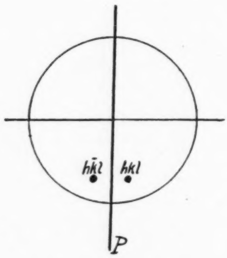

Fig. 20. Class 4.

5. Prismatic Class, $A_2 \cdot P \cdot C$.

4 operations: 1 , a_{180° , p , c .
(hkl) $(\bar{h}k\bar{l})$ $(h\bar{k}l)$ $(\bar{h}\bar{k}\bar{l})$

The " multiplication table " for this group is given on page 175.

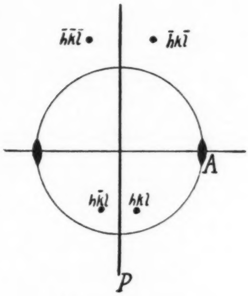

FIG. 21. Class 5.

ORTHORHOMBIC SYSTEM

6. RHOMBIC BISPHENOIDAL CLASS, $3A_2$.

4 operations: 1 , $a'_{180°}$, $a''_{180°}$, $a'''_{180°}$.
(hkl) $(\bar{h}\bar{k}l)$ $(h\bar{k}\bar{l})$ $(\bar{h}k\bar{l})$

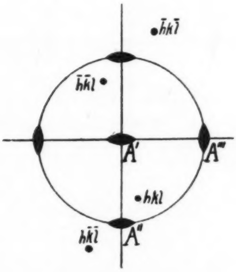

FIG. 22. Class 6.

7. RHOMBIC PYRAMIDAL CLASS, $A_2 \cdot 2P$.

4 operations: 1 , $a'_{180°}$, p'' , p'''.
(hkl) $(\bar{h}\bar{k}l)$ $(h\bar{k}l)$ $(\bar{h}kl)$

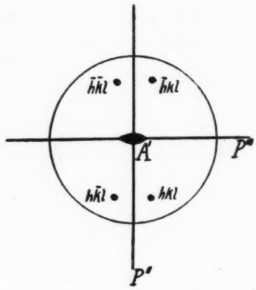

FIG. 23. Class 7.

8. RHOMBIC BIPYRAMIDAL CLASS, $3A_2 \cdot 3P \cdot C$.

8 operations:	1 ,	$a'_{180°}$,	$a''_{180°}$,	$a'''_{180°}$;	p' ,	p'' ,	p''',	c .
	(hkl)	$(\bar{h}\bar{k}l)$	$(h\bar{k}\bar{l})$	$(\bar{h}k\bar{l})$	$(hk\bar{l})$	$(h\bar{k}l)$	$(\bar{h}kl)$	$(\bar{h}\bar{k}\bar{l})$

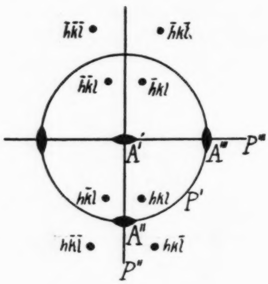

FIG. 24. Class 8.

TETRAGONAL SYSTEM.

9. TETRAGONAL BISPHENOIDAL CLASS, AP_4.

4 operations:	1 ,	$ap'_{90°}$,	$a'_{180°}$,	$ap'_{270°}$.
	(hkl)	$(\bar{k}h\bar{l})$	$(\bar{h}\bar{k}l)$	$(k\bar{h}\bar{l})$

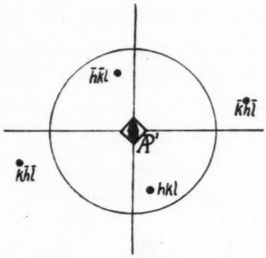

FIG. 25. Class 9.

10. TETRAGONAL PYRAMIDAL CLASS, A_4.

4 operations:	1 ,	$a'_{90°}$,	$a'_{180°}$,	$a'_{270°}$.
	(hkl)	$(\bar{k}hl)$	$(\bar{h}\bar{k}l)$	$(k\bar{h}l)$

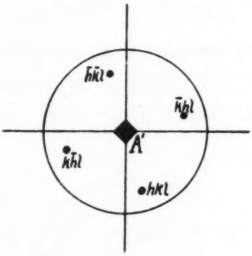

FIG. 26. Class 10.

11. TETRAGONAL SCALENOHEDRAL CLASS, $\bar{A}_4 \cdot 2A_2 \cdot 2P$.

8 operations:	1 ,	$ap'_{90°}$,	$a'_{180°}$,	$ap'_{270°}$;	$a''_{180°}$,	$a'''_{180°}$;	p^{IV},	p^{V}.
	(hkl)	$(\bar{k}h\bar{l})$	$(\bar{h}\bar{k}l)$	$(k\bar{h}\bar{l})$	$(h\bar{k}\bar{l})$	$(\bar{h}k\bar{l})$	(khl)	$(\bar{k}\bar{h}l)$

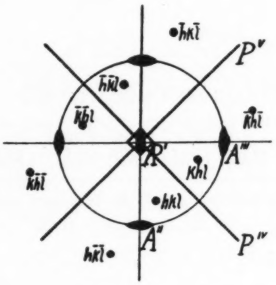

FIG. 27. Class 11.

12. Tetragonal Trapezohedral Class, $A_4 \cdot 4A_2$.

8 operations:	1,	$a'_{90°}$,	$a'_{180°}$,	$a'_{270°}$;	$a''_{180°}$,	$a'''_{180°}$,	$a^{IV}_{180°}$,	$a^{V}_{180°}$.
	(hkl)	$(\bar{k}hl)$	$(\bar{h}\bar{k}l)$	$(k\bar{h}l)$	$(k\bar{k}\bar{l})$	$(\bar{h}k\bar{l})$	$(hk\bar{l})$	$(\bar{h}\bar{k}l)$

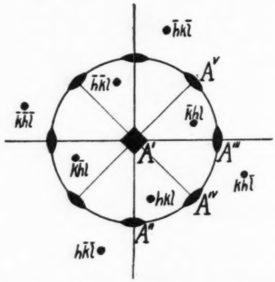

Fig. 28. Class 12.

13. Tetragonal Bipyramidal Class, $A_4[AP_4] \cdot P \cdot C$.[43]

8 operations:	1,	$a'_{90°}$,	$a'_{180°}$,	$a'_{270°}$,	$ap'_{90°}$,	$ap'_{270°}$,	p',	c
	(hkl)	$(k\bar{h}l)$	$(\bar{h}\bar{k}l)$	$(k\bar{h}l)$	$(\bar{k}h\bar{l})$	$(k\bar{h}\bar{l})$	$(hk\bar{l})$	$(\bar{h}\bar{k}\bar{l})$

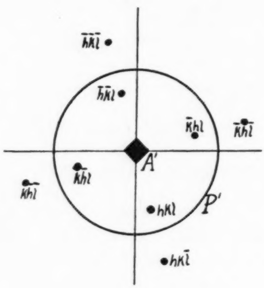

Fig. 29. Class 13.

14. Ditetragonal Pyramidal Class, $A_4 \cdot 4P$.

8 operations:	1,	$a'_{90°}$,	$a'_{180°}$,	$a'_{270°}$;	p'',	p''',	p^{IV},	p^{V}.
	(hkl)	$(\bar{k}hl)$	$(\bar{h}\bar{k}l)$	$(k\bar{h}l)$	$(h\bar{k}l)$	$(\bar{h}kl)$	(khl)	$(\bar{k}\bar{h}l)$

[43] Since AP_4 has an operation ($a'_{180°}$) in common with A_4 it is enclosed in square brackets.

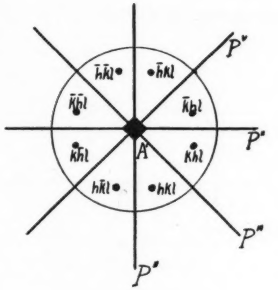

FIG. 30. Class 14.

15. DITETRAGONAL BIPYRAMIDAL CLASS, $A_4[AP_4] \cdot 4A_2 \cdot 5P \cdot C$.[43]

16 operations:

1 ,	$a'_{90°}$,	$a'_{180°}$,	$a'_{270°}$,	$ap'_{90°}$,	$ap'_{270°}$;	$a''_{180°}$,
(hkl)	$(\bar{k}h\bar{l})$	$(\bar{h}\bar{k}l)$	$(k\bar{h}l)$	$(\bar{k}h\bar{l})$	$(k\bar{h}\bar{l})$	$(h\bar{k}\bar{l})$

$a'''_{180°}$,	$a^{IV}_{180°}$,	$a^{V}_{180°}$;	p' ,	p'' ,	p''' ,	p^{IV} ,	p^{V} ,	c .
$(\bar{h}k\bar{l})$	$(kh\bar{l})$	$(\bar{k}\bar{h}\bar{l})$	$(hk\bar{l})$	$(h\bar{k}l)$	$(\bar{h}kl)$	$(\bar{k}hl)$	$(\bar{k}\bar{h}l)$	$(\bar{h}\bar{k}\bar{l})$

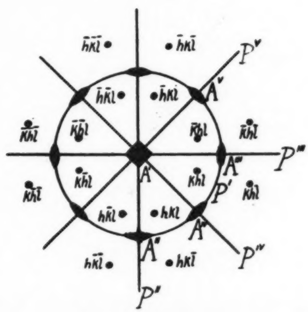

FIG. 31. Class 15.

HEXAGONAL SYSTEM.

RHOMBOHEDRAL SUBSYSTEM.

16. TRIGONAL PYRAMIDAL CLASS, A_3.

3 operations:

1 ,	$a'_{120°}$,	$a'_{240°}$.
$(hk\bar{i}l)$	$(\bar{i}hkl)$	$(k\bar{i}hl)$

This is the only prime group of the 32.

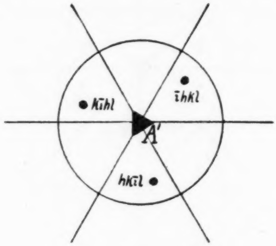

FIG. 32. Class 16.

17. RHOMBOHEDRAL CLASS, $AP_6(C)$.

$$\begin{array}{lcccccc} \text{6 operations:} & 1\;, & ap'_{60^\circ}, & a'_{120^\circ}\,, & c\;, & a_{240^\circ}, & ap'_{300^\circ}. \\ & (hk\bar{i}l) & (\bar{k}i\bar{h}\bar{l}) & (\bar{i}hkl) & (\bar{h}\bar{k}i\bar{l}) & (k\bar{i}hl) & (i\bar{h}\bar{k}\bar{l}) \end{array}$$

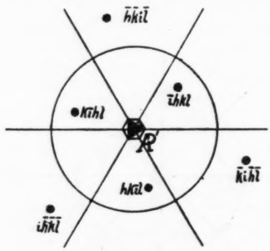

FIG. 33. Class 17.

Hilton states that the axis of symmetry in this class is a 3-fold rotatory-inversion axis, and overlooks the fact that rotatory-inversion axes are always of even period. Three powers of ca_{120° do not form a group; the six powers of ap_{60° are required. The principal axis in this class is a rotatory-reflection axis of 6-fold symmetry.

18. TRIGONAL TRAPEZOHEDRAL CLASS, $A_3 \cdot 3A_2$.

$$\begin{array}{lcccccc} \text{6 operations:} & 1\;, & a'_{120^\circ}\,, & a'_{240^\circ}\;; & a^{\mathrm{V}}_{180^\circ}\,, & a^{\mathrm{VI}}_{180^\circ}\,, & a^{\mathrm{VII}}_{180^\circ}. \\ & (hk\bar{i}l) & (\bar{i}hkl) & (k\bar{i}hl) & (kh\bar{i}\bar{l}) & (\bar{i}kh\bar{l}) & (h\bar{i}k\bar{l}) \end{array}$$

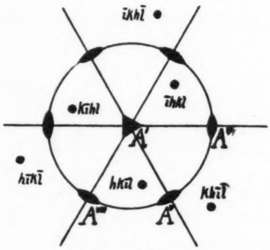

FIG. 34. Class 18.

19. DITRIGONAL PYRAMIDAL CLASS, $A_3 \cdot 3P$.

6 operations:	1 ,	$a'_{120°}$,	$a'_{240°}$;	p'' ,	p''' ,	p^{IV} .
	$(hk\bar{i}l)$	$(\bar{i}hkl)$	$(k\bar{i}hl)$	$(i\bar{k}\bar{h}l)$	$(\bar{h}i\bar{k}l)$	$(\bar{k}\bar{h}il)$

The " multiplication table " for this group is given on page 175.

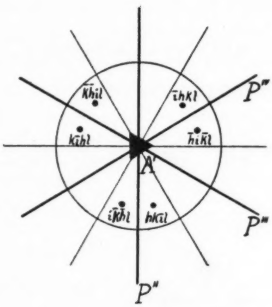

FIG. 35. Class 19.

20. HEXAGONAL SCALENOHEDRAL CLASS, $AP_6 \cdot 3A_2 \cdot 3P \cdot (C)$.

12 operations:	1 ,	$ap'_{60°}$,	$a'_{120°}$,	c ,	$a'_{240°}$,	$ap'_{300°}$;
	$(hk\bar{i}l)$	$(\bar{k}i\bar{h}\bar{l})$	$(\bar{i}hkl)$	$(\bar{h}\bar{k}i\bar{l})$	$(k\bar{i}hl)$	$(i\bar{h}\bar{k}\bar{l})$
	$a^{\mathrm{V}}_{180°}$,	$a^{\mathrm{VI}}_{180°}$,	$a^{\mathrm{VII}}_{180°}$;	p'' ,	p''' ,	p^{IV} .
	$(hk\bar{i}\bar{l})$	$(\bar{i}kh\bar{l})$	$(h\bar{i}k\bar{l})$	$(i\bar{k}\bar{h}l)$	$(\bar{h}i\bar{k}l)$	$(\bar{k}\bar{h}il)$

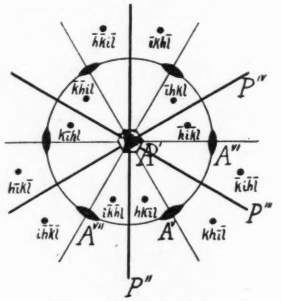

Fig. 36. Class 20.

Here again Hilton calls the principal axis a 3-fold axis of rotatory-inversion. Rotatory-inversion axes are always of even degree. The principal axis in this class is a 6-fold axis of rotatory-reflection. Unfortunately Hilton's mnemonic distinction between the hexagonal and rhombohedral subsystems fails.

HEXAGONAL SUBSYSTEM.

21. Trigonal Bipyramidal Class, $CA_6(P)$.

6 operations: 1 , ca'_{60° , a'_{120° , p' , a'_{240° , ca'_{300° .
$(hk\bar{i}l)$ $(k\bar{i}h\bar{l})$ $(\bar{i}hkl)$ $(hk\bar{i}\bar{l})$ $(k\bar{i}hl)$ $(\bar{i}hk\bar{l})$

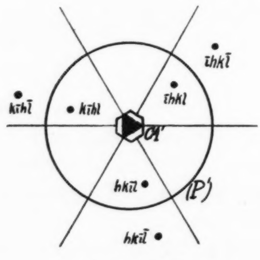

Fig. 37. Class 21.

The usual method of representing this group as $A_3 \cdot P$ is incomplete as six operations are involved instead of four.

Schoenflies and Hilton indicate the group as follows (my symbols are used):

$$\left\{ \begin{matrix} 1, & a'_{120°}, & a'_{240°} \\ p', & p' \cdot a'_{120°}, & p' \cdot a'_{240} \end{matrix} \right\}$$

Here, two of the operations are indicated as products; p is the fourth "generating" operation. If we indicate each operation by a distinct symbol ($\mathit{ca}'_{60°}$ instead of $p' \cdot a'_{240°}$ and $\mathit{ca}'_{300°}$ instead of $p' \cdot a'_{120°}$) and not part of them as products, it is obvious that we have a cyclic group. The group CA_6 is exactly analogous to AP_6; in the former case reflection is the third power of ca_{60}; in the latter, inversion is the third power of $\mathit{ap}_{60°}$.

Jaeger[44] has stated that the trigonal bipyramidal class has a 3-fold axis of rotatory-reflection (he uses the symbol $\bar{A}_3$). But since there are six distinct operations involved, the axis is clearly 6-fold and not 3-fold. Three powers of $\mathit{ap}_{120°}$ do not form a group; the six powers of $\mathit{ca}_{60°}$ are required. CA_6, or its equivalent, then, and not AP_3, or its equivalent, must be used to indicate the symmetry in this class. In the general symbol of an axis of rotatory-reflection AP_n, n is always an even number. With a series of rotatory-reflections it requires an even number of operations to come back to identity.

This class is interesting as it is the only class of the 32 for which no representative has yet been found.

22. Ditrigonal Bipyramidal Class, $\mathit{CA}_6(P) \cdot 3A_2 \cdot 3P$.

$$\begin{matrix} \text{12 operations:} & 1, & \mathit{ca}'_{60°}, & a'_{120°}, & p, & a'_{240°}, & \mathit{ca}'_{300°}; \\ & (hk\bar{i}l) & (k\bar{i}h\bar{l}) & (\bar{i}hkl) & (hk\bar{i}\bar{l}) & (k\bar{i}hl) & (\bar{i}hk\bar{l}) \\ & a''_{180°}, & a'''_{180°}, & a^{\mathrm{IV}}_{180°}; & p'', & p''', & p^{\mathrm{IV}}. \\ & (i\bar{k}\bar{h}\bar{l}) & (\bar{h}i\bar{k}\bar{l}) & (\bar{k}\bar{h}i\bar{l}) & (i\bar{k}\bar{h}l) & (\bar{h}i\bar{k}l) & (\bar{k}\bar{h}il) \end{matrix}$$

[44] Loc. cit., pp. 23–24.

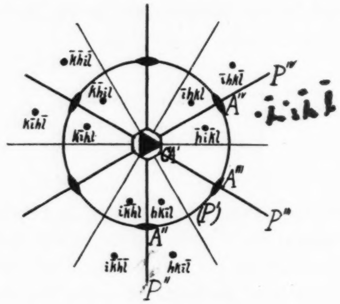

FIG. 38. Class 22.

This class is not, as Groth and others have concluded, the holosymmetric or holohedral division of a trigonal system, but belongs in the hexagonal system or subsystem. It is a subgroup (see p. 198) of class 27, but classes 17 and 20 are not subgroups of it.

23. HEXAGONAL PYRAMIDAL CLASS, A_6.

6 operations: 1, $a'_{60°}$, $a'_{120°}$, $a'_{180°}$, $a'_{240°}$, $a'_{300°}$.
$(hk\bar{i}l)$ $(\bar{k}i\bar{h}l)$ $(\bar{i}hkl)$ $(\bar{h}\bar{k}il)$ $(k\bar{i}hl)$ $(i\bar{h}\bar{k}l)$

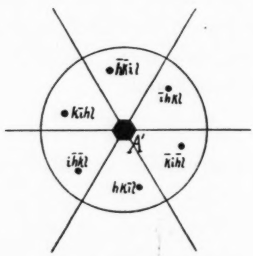

FIG. 39. Class 23.

24. HEXAGONAL TRAPEZOHEDRAL CLASS, $A_6 \cdot 6A_2$.

12 operations: 1, $a'_{60°}$, $a'_{120°}$, $a'_{180°}$, $a'_{240°}$, $a'_{300°}$,
$(hk\bar{i}l)$ $(\bar{k}i\bar{h}l)$ $(\bar{i}hkl)$ $(\bar{h}\bar{k}il)$ $(k\bar{i}hl)$ $(i\bar{h}\bar{k}l)$

$a''_{180°}$, $a'''_{180°}$, $a^{\mathrm{IV}}_{180°}$, $a^{\mathrm{V}}_{180°}$, $a^{\mathrm{VI}}_{180°}$, $a^{\mathrm{VII}}_{180°}$.
$(i\bar{h}\bar{k}l)$ $(\bar{h}ik\bar{l})$ $(\bar{k}\bar{h}i\bar{l})$ $(kh\bar{i}\bar{l})$ $(\bar{i}hk\bar{l})$ $(h\bar{i}k\bar{l})$

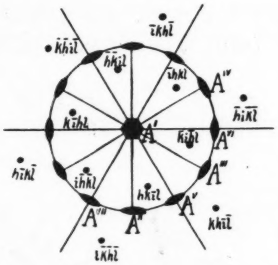

FIG. 40. Class 24.

25. HEXAGONAL BIPYRAMIDAL CLASS, $A_6\,[AP_6]\,[CA_6]\cdot(P)\cdot(C)$[45].

12 operations:	1 ,	$a'_{60°}$,	$a'_{120°}$,	$a'_{180°}$,	$a'_{240°}$,	$a'_{300°}$;
	$(hk\bar{i}l)$	$(\bar{k}i\bar{h}l)$	$(\bar{i}hkl)$	$(\bar{h}\bar{k}il)$	$(k\bar{i}hl)$	$(i\bar{k}\bar{h}l)$
	$ap'_{60°}$,	c ,	$ap'_{300°}$;	$ca'_{60°}$,	p' ,	$ca'_{300°}$.
	$(\bar{k}i\bar{h}\bar{l})$	$(\bar{h}\bar{k}i\bar{l})$	$(i\bar{h}\bar{k}\bar{l})$	$(k\bar{i}h\bar{l})$	$(hk\bar{i}\bar{l})$	$(\bar{i}hk\bar{l})$

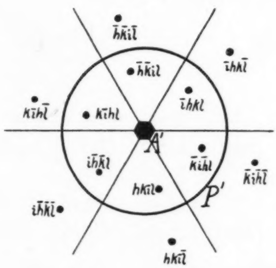

FIG. 41. Class 25.

This group is usually indicated by the equivalent of the symbol $A_6\cdot P\cdot C$, but this expression is incomplete as it includes eight operations instead of twelve. The axis of 6-fold symmetry is also an axis of rotatory-reflection and one of rotatory-inversion.

26. DIHEXAGONAL PYRAMIDAL CLASS, $A_6\cdot 6P$.

$$
\begin{array}{lcccccc}
\text{12 operations:} & 1, & a'_{60^\circ}, & a'_{120^\circ}, & a'_{180^\circ}, & a'_{240^\circ}, & a'_{300^\circ}; \\
 & (hk\bar{i}l) & (\bar{k}i\bar{h}l) & (\bar{i}hkl) & (\bar{h}\bar{k}il) & (k\bar{i}hl) & (i\bar{h}\bar{k}l) \\
 & p'', & p''', & p^{\mathrm{IV}}, & p^{\mathrm{V}}, & p^{\mathrm{VI}}, & p^{\mathrm{VII}}. \\
 & (i\bar{k}\bar{h}l) & (\bar{h}i\bar{k}l) & (\bar{k}\bar{h}il) & (kh\bar{i}l) & (\bar{i}khl) & (h\bar{i}kl)
\end{array}
$$

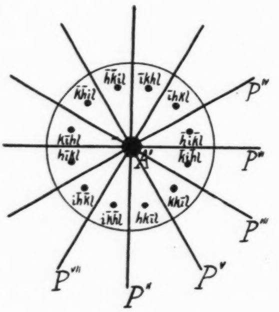

FIG. 42. Class 26.

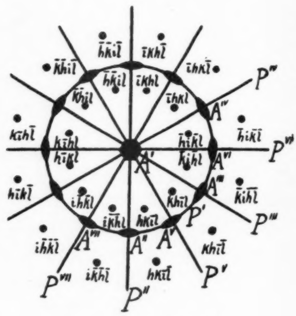

FIG. 43. Class 27.

27. DIHEXAGONAL BIPYRAMIDAL CLASS,

$A_6 [AP_6] [CA_6] \cdot 6A_2 \cdot 6P \cdot (P) \cdot (C)$.[45]

$$
\begin{array}{lcccccc}
\text{24 operations:} & 1, & a'_{60^\circ}, & a'_{120^\circ}, & a'_{180^\circ}, & a'_{240^\circ}, & a'_{300^\circ}; \\
 & (hk\bar{i}l) & (\bar{k}i\bar{h}l) & (\bar{i}hkl) & (\bar{h}\bar{k}il) & (k\bar{i}hl) & (i\bar{h}\bar{k}l)
\end{array}
$$

$$
\begin{array}{ccccccc}
ap'_{60^\circ}, & c, & ap'_{300^\circ}; & ca'_{60^\circ}, & p', & ca'_{300^\circ}; & a''_{180^\circ}, \\
(\bar{k}i\bar{h}\bar{l}) & (\bar{h}\bar{k}i\bar{l}) & (i\bar{h}\bar{k}\bar{l}) & (k\bar{i}h\bar{l}) & (hk\bar{i}\bar{l}) & (\bar{i}hk\bar{l}) & (i\bar{k}\bar{h}\bar{l}) \\
a'''_{180^\circ}, & a^{\mathrm{IV}}_{180^\circ}, & a^{\mathrm{V}}_{180^\circ}, & a^{\mathrm{VI}}_{180^\circ}, & a^{\mathrm{VII}}; & p'', & p''', \\
(\bar{h}i\bar{k}\bar{l}) & (\bar{k}\bar{h}i\bar{l}) & (kh\bar{i}\bar{l}) & (\bar{i}kh\bar{l}) & (h\bar{i}k\bar{l}) & (i\bar{k}\bar{h}l) & (\bar{h}i\bar{k}l) \\
p^{\mathrm{IV}}, & p^{\mathrm{V}}, & p^{\mathrm{VI}}, & p^{\mathrm{VII}}. & & & \\
(\bar{k}\bar{h}il) & (hk\bar{i}l) & (\bar{i}hkl) & (h\bar{i}kl) & & &
\end{array}
$$

---

[45] In groups 25 and 27, A_6, AP_6, and CA_6, each have two operations (a'_{120° and a'_{240°) in common, hence square brackets are placed around AP_6 and CA_6.

As in class 25, the principal axis is an axis of rotatory-reflection and rotatory-inversion as well as an ordinary 6-fold axis.

ISOMETRIC SYSTEM.

28. Tetartoidal Class, $4A_3 \cdot 3A_2$.

$$\begin{array}{lllllll} \text{12 operations:} & 1\ , & a'_{120^\circ}, & a'_{240}, & a''_{120^\circ}, & a''_{240^\circ}; & a'''_{120^\circ}, \\ & (hkl) & (lhk) & (klh) & (\bar{h}l\bar{h}) & (\bar{l}\bar{h}k) & (\bar{l}h\bar{k}) \end{array}$$

$$\begin{array}{llllll} a'''_{240^\circ}; & a^{\mathrm{IV}}_{120^\circ}, & a^{\mathrm{IV}}_{240^\circ}; & a'_{180^\circ}, & a''_{180^\circ}, & a'''_{180^\circ}. \\ (k\bar{l}\bar{h}) & (\bar{k}\bar{l}h) & (l\bar{h}\bar{k}) & (\bar{h}\bar{k}l) & (h\bar{k}\bar{l}) & (\bar{h}k\bar{l}) \end{array}$$

This is the group of "tetrahedral rotations."

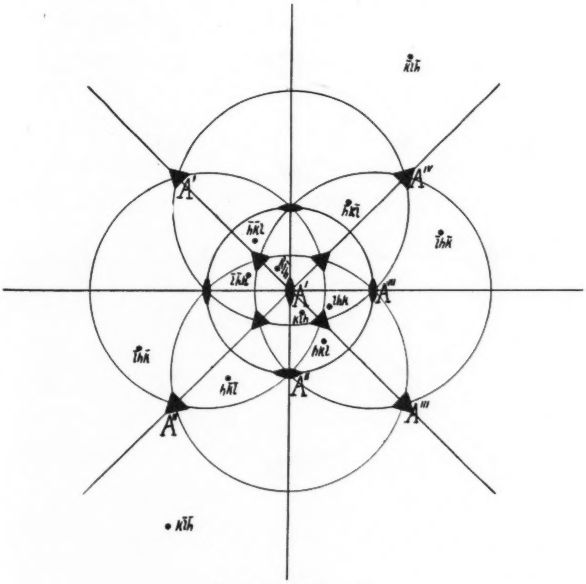

Fig. 44. Class 28.

29. Gyroidal Class, $3A_4 \cdot 4A_3 \cdot 6A_2$.

$$
\begin{array}{llllllll}
\text{24 operations:} & 1\ , & a'_{90°}, & a'_{180°}, & a'_{270°}; & a''_{90°}, & a''_{180°}, & a''_{270°}; \\
 & (hkl) & (\bar{k}hl) & (\bar{h}\bar{k}l) & (k\bar{h}l) & (h\bar{l}k) & (h\bar{k}\bar{l}) & (hl\bar{k})
\end{array}
$$

$$
\begin{array}{llllllll}
a'''_{90°}, & a'''_{180°}, & a'''_{270°}; & a'_{120°}, & a'_{240°}; & a''_{120°}, & a''_{240°}; & a'''_{180°}, \\
(lk\bar{h}) & (\bar{h}k\bar{l}) & (\bar{l}kh) & (lhk) & (klh) & (\bar{k}l\bar{h}) & (\bar{l}\bar{h}k) & (\bar{l}h\bar{k})
\end{array}
$$

$$
\begin{array}{lllllllll}
a'''_{240°}; & a^{\mathrm{IV}}_{120°}, & a^{\mathrm{IV}}_{240°}; & a^{\mathrm{IV}}_{180°}, & a^{\mathrm{V}}_{180°}, & a^{\mathrm{VI}}_{180°}, & a^{\mathrm{VII}}_{180°}, & a^{\mathrm{VIII}}_{180°}, & a^{\mathrm{IX}}_{180°}. \\
(k\bar{l}\bar{h}) & (\bar{k}\bar{l}h) & (l\bar{h}\bar{k}) & (kh\bar{l}) & (\bar{k}\bar{h}\bar{l}) & (\bar{l}\bar{k}\bar{h}) & (\bar{h}\bar{l}\bar{k}) & (l\bar{k}h) & (\bar{h}lk)
\end{array}
$$

This is the group of "octahedral rotations."

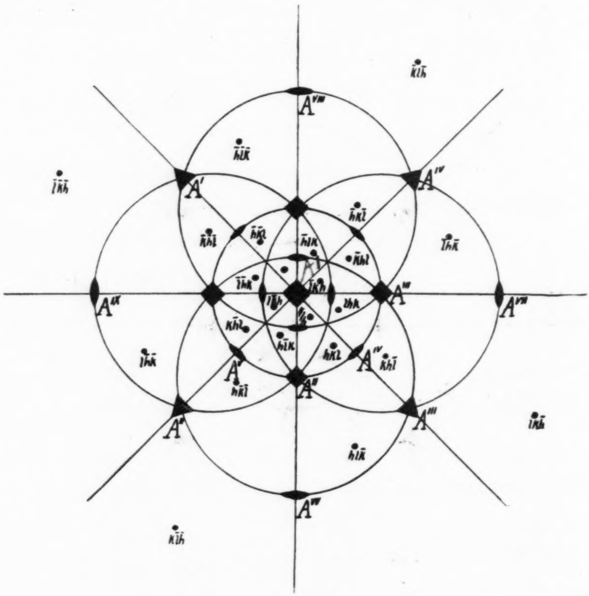

Fig. 45. Class 29.

30. Diploidal Class, $4\mathit{A\!\!\!/}P_6 \cdot 3A_2 \cdot 3P \cdot (C)$.

24 operations:

1 ,	$ap'_{60°}$,	$a'_{120°}$,	c ,	$a'_{240°}$,	$ap'_{300°}$;	$ap''_{60°}$,		
(hkl)	$(\bar{k}\bar{l}\bar{h})$	(lhk)	$(\bar{h}\bar{k}\bar{l})$	(klh)	$(\bar{l}\bar{h}\bar{k})$	$(lh\bar{k})$		
$a''_{120°}$,	$a''_{240°}$,	$ap''_{300°}$,	$ap'''_{60°}$,	$a'''_{120°}$,	$a'''_{240°}$,	$ap'''_{300°}$,	$ap^{\text{IV}}_{60°}$,	
$(\bar{k}l\bar{h})$	$(\bar{l}\bar{h}k)$	$(k\bar{l}h)$	$(\bar{k}lh)$	$(\bar{l}h\bar{k})$	$(k\bar{l}\bar{h})$	$(l\bar{h}k)$	$(\bar{l}hk)$	
$a^{\text{IV}}_{120°}$,	$a^{\text{IV}}_{240°}$,	$ap^{\text{IV}}_{300°}$;	$a'_{180°}$,	$a''_{180°}$,	$a'''_{180°}$;	p' ,	p'',	p'''.
$(\bar{k}\bar{l}h)$	$(l\bar{h}\bar{k})$	$(kl\bar{h})$	$(\bar{h}\bar{k}l)$	$(h\bar{k}\bar{l})$	$(\bar{h}k\bar{l})$	$(hk\bar{l})$	$(h\bar{k}l)$	$(\bar{h}kl)$

In this class and in class 32 there are $4ap_{60°}$, but the third power of each of these operations gives the same face $(\bar{h}\bar{k}\bar{l})$ by inversion, which is additional proof that the center of symmetry is a true element of symmetry.

This is the "extended group of tetrahedral rotations."

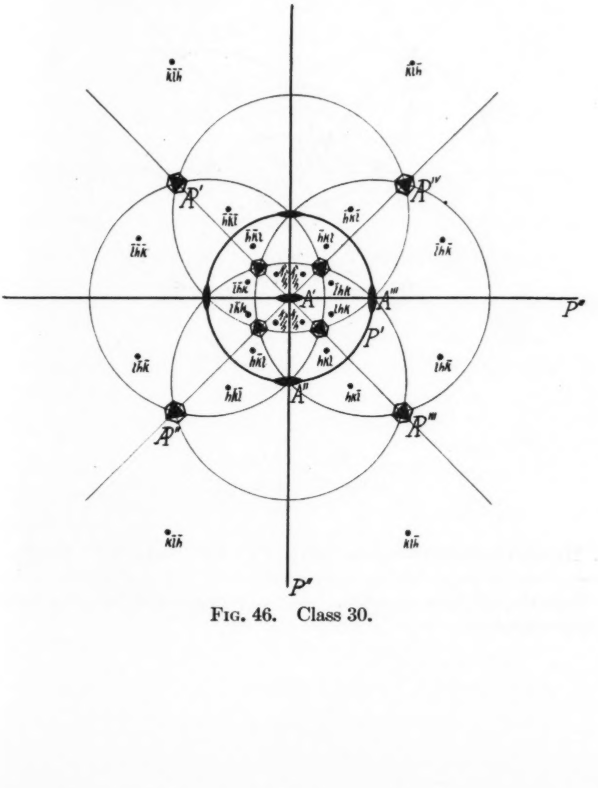

Fig. 46. Class 30.

31. Hextetrahedral Class, $3AP_4 \cdot 4A_3 \cdot 6P$.

24 operations:

1,	$ap'_{90°}$,	$a'_{180°}$,	$ap'_{270°}$;	$ap''_{90°}$,	$a''_{180°}$,	$ap''_{270°}$;
(hkl)	$(\bar{k}h\bar{l})$	$(\bar{h}\bar{k}l)$	$(k\bar{h}\bar{l})$	$(\bar{h}\bar{l}k)$	$(h\bar{k}\bar{l})$	$(\bar{h}l\bar{k})$

$ap'''_{90°}$,	$a'''_{180°}$,	$ap'''_{270°}$;	$a'_{120°}$,	$a'_{240°}$;	$a''_{120°}$,	$a''_{240°}$;	$a'''_{120°}$,
$(l\bar{k}\bar{h})$	$(\bar{h}k\bar{l})$	$(\bar{l}\bar{k}h)$	(lhk)	(klh)	$(\bar{k}l\bar{h})$	$(\bar{l}\bar{h}k)$	$(\bar{l}h\bar{k})$

$a'''_{240°}$;	$a^{IV}_{120°}$,	$a^{IV}_{240°}$;	p^{IV},	p^{V},	p^{VI},	p^{VII},	p^{VIII},	p^{IX}.
$(k\bar{l}\bar{h})$	$(\bar{k}\bar{l}h)$	$(l\bar{h}\bar{k})$	(khl)	$(\bar{k}\bar{h}l)$	(hlk)	(lkh)	$(h\bar{l}\bar{k})$	$(\bar{l}k\bar{h})$

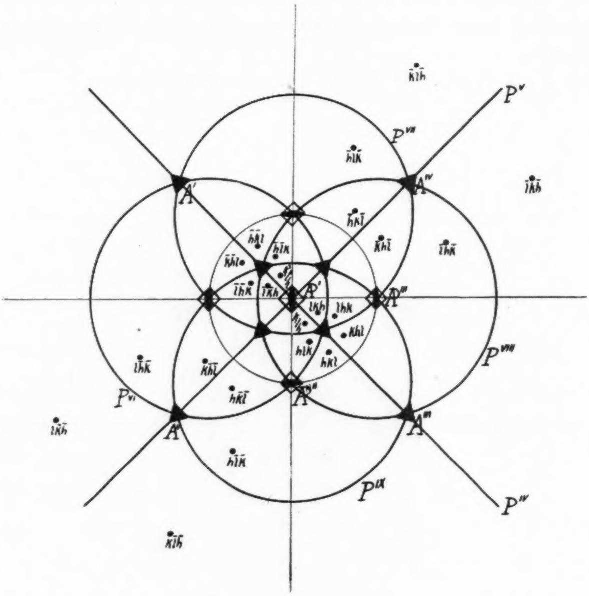

Fig. 47. Class 31.

32. Hexoctahedral Class, $3A_4[3AP_4] \cdot 4AP_6 \cdot 6A_2 \cdot 9P \cdot (C)$.[46]

[46] Since the $3AP_4$ have operations ($a_{180°}$) in common with $3A_4$, they are placed in square brackets.

48 operations:

1, (hkl)	$a'_{90°}$, $(\bar{k}hl)$	$a'_{180°}$, $(\bar{h}\bar{k}l)$	$a'_{270°}$, $(k\bar{h}l)$	$ap'_{90°}$, $(\bar{k}h\bar{l})$	$ap'_{270°}$; $(k\bar{h}\bar{l})$		
$a''_{90°}$, $(h\bar{l}k)$	$a''_{180°}$, $(h\bar{k}\bar{l})$	$a''_{270°}$, $(hl\bar{k})$	$ap''_{90°}$, $(\bar{h}\bar{l}k)$	$ap''_{270°}$, $(\bar{h}l\bar{k})$	$a'''_{90°}$, $(lk\bar{h})$	$a'''_{180°}$, $(\bar{h}k\bar{l})$	$a'''_{270°}$, $(\bar{l}kh)$
$ap'''_{90°}$, $(l\bar{k}\bar{h})$	$ap'''_{270°}$, $(\bar{l}\bar{k}h)$	$ap'_{60°}$, $(\bar{k}\bar{l}\bar{h})$	$a'_{120°}$, (lhk)	c, $(\bar{h}\bar{k}\bar{l})$	$a'_{240°}$, (klh)	ap'_{300}, $(\bar{l}\bar{h}\bar{k})$	$ap''_{60°}$, $(lh\bar{k})$
$a''_{120°}$, $(\bar{k}l\bar{h})$	$a''_{240°}$, $(\bar{l}\bar{h}k)$	$ap''_{300°}$, $(k\bar{l}h)$	$ap'''_{60°}$, $(\bar{k}lh)$	$a'''_{120°}$, $(\bar{l}h\bar{k})$	$a'''_{240°}$, $(k\bar{l}\bar{h})$	$ap'''_{300°}$, $(l\bar{h}k)$	$ap^{IV}_{60°}$, $(\bar{l}hk)$
$a^{IV}_{120°}$, $(\bar{k}\bar{l}h)$	$a^{IV}_{240°}$, $(l\bar{h}\bar{k})$	$ap^{IV}_{300°}$, $(kl\bar{h})$	$a^{IV}_{180°}$, $(kh\bar{l})$	$a^{V}_{180°}$, $(\bar{k}\bar{h}\bar{l})$	$a^{VI}_{180°}$, $(\bar{l}k\bar{h})$	$a^{VII}_{180°}$, $(\bar{h}\bar{l}\bar{k})$	$a^{VIII}_{180°}$, $(l\bar{k}h)$
$a^{IX}_{180°}$, $(\bar{h}lk)$	p', $(hk\bar{l})$	p'', $(h\bar{k}l)$	p''', $(\bar{h}kl)$	p^{IV}, (khl)	p^{V}, $(\bar{k}\bar{h}l)$	p^{VI}, (hlk)	p^{VII}, (lkh)
p^{VIII}, $(h\bar{l}\bar{k})$	p^{IX}. $(\bar{l}k\bar{h})$						

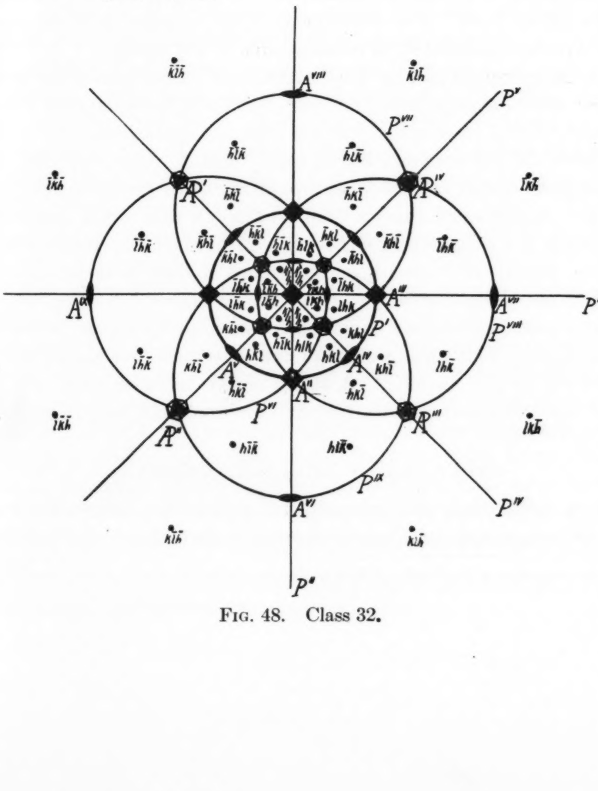

FIG. 48. Class 32.

See remark under Class 30.

This is the "extended group of octahedral rotations."

It has now been demonstrated that each face of the general form in any crystal class may be considered to be the result of some operation performed on the initial face. It is not necessary to use the product of two operations in order to derive a given face as is the usual custom. It is this fact that enables us to determine what the real symmetry operations are.

Subgroups.

If a certain portion of the operations of a group taken alone form a group, these operations constitute a *subgroup*. For example, in the group $\{1, a_{60^\circ}, a_{120^\circ}, a_{180^\circ}, a_{240^\circ}, a_{300^\circ}\}$ 1, a_{120°, and a_{240° form a subgroup; the same is also true of 1 and a_{180°. The number of operations of a subgroup is always an aliquot part of the number of operations of the group itself. For convenience, both identity and the group itself are here considered to be subgroups of any group.

The subgroups of each of the 32 groups or crystal classes are shown in the table on page 200. Apparently this is the first time that a complete list of these subgroups has ever been published.

Classes 27 (dihexagonal bipyramidal) and 32 (hexoctahedral) are unique in that they are not subgroups of any other group except themselves. At the same time it should be noted that every one of the 32 classes is a subgroup of either class 27 or class 32; classes 1–8 inclusive, and 16–20 inclusive are subgroups of both of them.

With the present arrangement of the 32 classes (it is not true of Groth's arrangement) the subgroups of each group appear before the group itself, This arrangement of the 32 classes first appeared in a text-book by the author.[47] A study of the table on page 200 will show that the sequence of the classes is only in minor part an arbitrary one.

Group Theory and Crystal Systems.

Since hemihedral and tetartohedral groups are subgroups of the holohedral groups, hemihedrism seems at first glance to have some

[47] *Introduction to the Study of Minerals and Rocks*, 2nd ed., p. 80, New York, 1921.

sanction from the standpoint of group theory.[48] It should be recognized, however, that the systems were chosen first, and so consideration should be given to all the subgroups of the various groups. It is possible to choose other groups as holohedral groups of systems which would have corresponding hemihedral groups. For example, classes 1 to 7 inclusive are subgroups of class 8. We might, then, place classes 1–8 inclusive in one system and consider classes 1–7 inclusive as hemihedral, tetartohedral, and ogdohedral divisions. This illustration serves to show that group theory of itself has no particular bearing on the validity of hemihedrism.

The question of holohedrism and hemihedrism seems to rest almost entirely upon whether or not the crystal systems are fundamental. As it must be admitted that crystal classes and not crystal systems are fundamental, the concept of holohedrism and hemihedrism loses its significance. The systems are largely a matter of convenience.

One point, however, is settled by means of group theory, and that is the separation of the hexagonal system into two divisions. The trigonal pyramidal, rhombohedral, trigonal trapezohedral, and hexagonal scalenohedral classes constitute the rhombohedral subsystem (or system). The trigonal bipyramidal and ditrigonal bipyramidal classes must be placed in the hexagonal subsystem (or system) and not in a trigonal or rhombohedral subsystem (or system) as has been done by Groth, Dana, Tutton, Moses, Moses and Parsons, Lewis, Swartz, Wülfing, Niggli, Jaeger, Cole, Spencer, Rinne, and others. Classes 16 to 19 inclusive (see table p. 200) are subgroups of class 20, but class 20 is not a subgroup of class 22.

Group theory has no bearing on the question whether classes 16 to 27 inclusive should be placed in one system or in two.

The inclusion of the five classes enumerated, and not the seven given by Groth, in a subsystem of the hexagonal system (or the rhombohedral system), is confirmed by crystal structure theory. Crystals of the five classes 16 to 20 inclusive have as a space lattice either a rhombohedron or a hexagonal prism. Crystals of the other seven classes, 21 to 27 inclusive, have only a hexagonal prism as a space lattice. The Laue diagrams of crystals of classes 21 and 22 are similar to those of classes 23 to 27 inclusive and not to those of classes 16 to 20 inclusive.

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[48] See Hilton, *Mathematical Crystallography*, p. 92.

TABLE OF THE 32 CRYSTAL CLASSES WITH THEIR SUBGROUPS

System		No.	Name of Class	Order	Symmetry	Subgroups
Triclinic		1	Asymmetric	1	None	1.
		2	Pinakoidal	2	C	1, 2.
Monoclinic		3	Sphenoidal	2	A_2	1, 3.
		4	Domatic	2	P	1, 4.
		5	Prismatic	4	$A_2 \cdot P \cdot C$	1, 2, 3, 4, 5.
Orthorhombic		6	Rhombic bisphenoidal	4	$3A_2$	1, 3, 6.
		7	" pyramidal	4	$A_2 \cdot 2P$	1, 3, 4, 7.
		8	" bipyramidal	8	$3A_2 \cdot 3P \cdot C$	1, 2, 3, 4, 5, 6, 7, 8.
Tetragonal		9	Tetragonal bisphenoidal	4	Æ_4	1, 3, 9.
		10	" pyramidal	4	A_4	1, 3, 10.
		11	" scalenohedral	8	$\text{Æ}_4 \cdot 2A_2 \cdot 2P$	1, 3, 4, 6, 7, 9, 11.
		12	" trapezohedral	8	$A_4 \cdot 4A_2$	1, 3, 6, 10, 12.
		13	" bipyramidal	8	$A_4[\text{Æ}_4] \cdot P \cdot C$	1, 2, 3, 4, 5, 9, 10, 13.
		14	Ditetragonal pyramidal	8	$A_4 \cdot 4P$	1, 3, 4, 7, 10, 14.
		15	" bipyramidal	16	$A_4[\text{Æ}_4] \cdot 4A_2 \cdot 5P \cdot C$	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
Hexagonal	Rhombohedral Subsystem	16	Trigonal pyramidal	3	A_3	1, 16.
		17	Rhombohedral	6	$\text{Æ}_6(C)$	1, 2, 16, 17.
		18	Trigonal trapezohedral	6	$A_3 \cdot 3A_2$	1, 3, 16, 18.
		19	Ditrigonal pyramidal	6	$A_3 \cdot 3P$	1, 4, 16, 19.
		20	Hexagonal scalenohedral	12	$\text{Æ}_6 \cdot 3A_2 \cdot 3P \cdot (C)$	1, 2, 3, 4, 5, 16, 17, 18, 19, 20.
	Hexagonal Subsystem	21	Trigonal bipyramidal	6	$CA_6(P)$	1, 4, 16, 21.
		22	Ditrigonal bipyramidal	12	$CA_6(P) \cdot 3A_2 \cdot 3P$	1, 3, 4, 7, 16, 18, 19, 21, 22.
		23	Hexagonal pyramidal	6	A_6	1, 3, 16, 23.
		24	" trapezohedral	12	$A_6 \cdot 6A_2$	1, 3, 6, 16, 18, 23, 24.
		25	" bipyramidal	12	$A_6[\text{Æ}_6][CA_6](P) \cdot (C)$	1, 2, 3, 4, 5, 16, 17, 21, 23, 25.
		26	Dihexagonal pyramidal	12	$A_6 \cdot 6P$	1, 3, 4, 7, 16, 19, 23, 26.
		27	" bipyramidal	24	$A_6[\text{Æ}_6][CA_6](P) \cdot 6A_2 \cdot 6P \cdot (C)$	1, 2, 3, 4, 5, 6, 7, 8, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27.
Isometric		28	Tetartoidal	12	$4A_3 \cdot 3A_2$	1, 3, 6, 16, 28.
		29	Gyroidal	24	$3A_4 \cdot 4A_3 \cdot 6A_2$	1, 3, 6, 10 12, 16, 18, 28, 29.
		30	Diploidal	24	$4\text{Æ}_6 \cdot 3A_2 \cdot 3P \cdot (C)$	1, 2, 3, 4, 5, 6, 7, 8, 16, 17, 28, 30.
		31	Hextetrahedral	24	$3\text{Æ}_4 \cdot 4A_3 \cdot 6P$	1, 3, 4, 6, 7, 9, 11, 16, 19, 28, 31.
		32	Hexoctahedral	48	$3A_4[3\text{Æ}_4] \cdot 4\text{Æ}_6 \cdot 6A_2 \cdot 9P \cdot (C)$	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 28, 29, 30, 31, 32

Subgroups found in two or more groups are known as common subgroups. The common subgroup of the highest order is called the *greatest common subgroup.*

Two of the six systems may be defined in terms of the greatest common subgroup. In the isometric system the greatest common subgroup is group 28 with the symmetry $4A_3 \cdot 3A_2$. In the hexagonal system the greatest common subgroup is group 16 with the symmetry A_3. But for the other systems this scheme fails. Group 3 (A_2), for example, is the greatest common subgroup for both the orthorhombic and tetragonal systems. Neither the monoclinic nor the triclinic system has a common subgroup (except identity). It is evident, then, that the theory of groups is not a sufficient guide for the establishment of crystal systems, although it may be of some assistance.

Summary.

The symmetry operations of a crystal are the operations that are necessary to derive each and every face of the general form from an initial face.

The symmetry operations possible in crystals are the following:

Identical operation: 1.

Rotations about an axis: a_{60°, a_{90°, a_{120°, a_{180°, a_{240°, a_{270°, a_{300°.

Reflection in a plane : p.

Inversion about the center: c.

Rotatory-reflections: ap_{60°, ap_{90°, ap_{270°, ap_{300°.

Rotatory-inversions: ca_{60°, ca_{300°.

All of these symmetry operations are represented by faces of the general form on crystals of either beryl (dihexagonal bipyramidal class) or garnet (hexoctahedral class).

Counting the various positions on a crystal, of axes, planes, rotatory-reflection axes, and rotatory-inversion axes of symmetry, there are, in all, 64 symmetry operations found on crystals.

Since inversion is a single operation, it should be used instead of a rotatory-reflection of 180°. If there is one 2-fold axis of rotatory-reflection, there must be an infinite number of such axes.

Both rotatory-inversions and rotatory-reflections must be used as symmetry operations.

The terms *rotoflection* and *rotoversion* may be used for rotatory-reflection and rotatory-inversion respectively.

The following are the elements of symmetry possible in crystals:

Axes of symmetry: A_2, A_3, A_4, A_6.

Plane of symmetry: P.

Center of symmetry: C.

Rotatory-reflection axis-plane: AP_4, AP_6.

Rotatory-inversion axis-center: CA_6.

Each of the elements of symmetry implies the existence of n symmetry operations which are powers of a single operation.

Each of the above elements of symmetry has actually been observed on crystals.

The mathematical theory of groups may be applied to the study of crystal symmetry. The symmetry operations of a crystal form a finite group. The group may be expressed by means of symmetry operations or symmetry elements. Each of the 32 crystal classes constitutes a distinct group. Each of the elements of symmetry taken by itself represents a cyclic group. Of the 32 groups nine are cyclic groups. The symmetry operations and symmetry elements of each of the 32 groups have been determined. The use of both rotatory-reflections and rotatory-inversions reconciles our usual conception of symmetry with group theory.

The trigonal bipyramidal class (No. 21) should be represented by the symmetry element CA_6 and not by $A_3 \cdot P$, for it is a cyclic group formed by the various powers of the operation $ca_{60°}$. CA_6 is also a symmetry element in the ditrigonal bipyramidal (No. 22), hexagonal bipyramidal (No. 25), and dihexagonal bipyramidal (No. 27) classes. The hexagonal bipyramidal (No. 25) and dihexagonal bipyramidal (No. 27) classes include as symmetry elements A_6, AP_6, and CA_6.

In the diploidal (No. 30) and hexoctahedral (No. 32) classes the symmetry elements include $4AP_6$; here inversion is an operation (the third power of $ap_{60°}$) common to each of these four rotatory-reflection axes. This is additional evidence that the center of symmetry is a true element of symmetry.

The tetragonal bipyramidal (No. 13), ditetragonal bipyramidal (No. 15), and hexoctahedral (No. 32) classes include as symmetry elements both A_4 and AP_4.

In the tabulation of the 32 crystal classes (p. 200), the order of Groth has been followed except that the trigonal bipyramidal (No. 21) class has been placed after the hexagonal scalenohedral (No. 20) class.

In this tabulation all the subgroups of each of the 32 groups have been determined for the first time, apparently. As now arranged, the subgroups of each group appear in the tabulation before the group itself; so that the sequence of the classes is only in minor part arbitrary. Numbers, then, as well as names may be used for the 32 crystal classes.

Although the so-called hemihedral groups are subgroups of the so-called holohedral groups, the idea of hemihedrism receives no particular sanction from the theory of groups.

Although the classes of the hexagonal and isometric systems each have a greatest common subgroup, group theory is not a sufficient guide for the establishment of the crystal systems. Group theory, however, does show that the five classes Nos. 16 to 20 inclusive and not the seven classes Nos. 16 to 22 inclusive should be included in the rhombohedral subsystem (or system in case seven systems are used).

In conclusion it may be stated that an analytical study of crystal symmetry with the aid of the theory of groups establishes definitely just what constitues the symmetry operations and symmetry elements of crystals. The subject of crystal symmetry would now seem to have reached something like finality by reason of its mathematical nature.

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